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The photogenic Cauchy transform

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Abstract

In this paper the Dirac operator on the Klein model for the hyperbolic unit ball is considered and a Cauchy-type integral transform is defined, by means of a Cauchy kernel with singularity on the nullcone.
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1. Introduction

In recent papers the authors have developed a function theory for the Dirac operator on the hyperbolic unit ball—realized as the manifold of rays inside the future cone FC in the flat Minkowski space–time $\mathbb{R}^{1,m}$ (see [9,10])—within the framework of Clifford analysis, a direct and elegant generalization to higher dimension of the theory of holomorphic functions in the complex plane. Whereas most of the classical literature concerns the Dirac operator on the flat Euclidean space (see [1,7,13]), a natural generalization consists in studying the Dirac operator on general manifolds within the framework of Clifford analysis (see [2,5]). For the particular choice of a positively curved Riemannian manifold we refer to the work of Liu, Ryan, Van Lancker and Sommen (see [17,18,20]), whereas in this paper we consider

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a model for the negatively curved Riemannian space. The nature of our model is *projective*, hereby inspired by Gel'fand and collaborators (see [11]), whence all relevant objects are to be defined on the manifold of rays. In particular, the fundamental solution for the Dirac operator on the hyperbolic unit ball was defined on the manifold of rays by considering a homogenous Clifford line bundle and by defining functions on the hyperbolic unit ball as sections of this bundle (see [3,4,8]). This means that functions on the hyperbolic unit ball are *homogeneous* in space–time coordinates $(T, \underline{X})$ and that the Dirac operator on the hyperbolic unit ball is defined as the Dirac operator on the flat Minkowski space–time $\mathbb{R}^{1,m}$ acting on homogeneous functions.

The fundamental solution for the hyperbolic Dirac equation, as constructed in Ref. [9], is α-homogeneous in space–time coordinates (α being a complex number) and becomes singular on the positive time-axis. In order to develop a function theory on the hyperbolic unit ball, using integral formulae such as Stokes' and Cauchy's theorem, these singularities were to be removed from the time-axis to an arbitrary ray inside the future cone in such a way that the transformations involved commute with the invariance group of the Dirac operator, which is the group $\mathrm{Spin}(1, m)$. This means that we have used a *pure* boost to remove the singularities to an arbitrary ray inside FC.

However, the positive time-axis cannot be boosted to a ray belonging to the upper part of the nullcone, as this would (relativistically speaking) require an infinite amount of energy. Nevertheless, in this paper we solve the equation for the fundamental solution of the Dirac operator on the hyperbolic unit ball having a singularity on the nullcone, by introducing *manu militare* a delta function *on* the nullcone.

This will be done by means of Clifford analysis techniques, for which we refer to Section 2, and Riesz' distributions, introduced in Section 3. The so-called photogenic fundamental solution will be constructed in Section 4 and in Section 5 it will be used to define the photogenic Cauchy transform. In Section 6 its boundary values will be determined, yielding results for the space $L_2(S^{m-1})$ of square integrable functions on the unit sphere $S^{m-1} \subset \mathbb{R}^m$.

2. Clifford algebras and Clifford analysis

2.1. Clifford analysis on flat Euclidean space

Consider an orthonormal basis $\{e_1, \ldots, e_m\}$ for $\mathbb{R}^m$ endowed with the standard Euclidean inner product $< \underline{x}, \underline{y} >= \sum_j x_j y_j$. The Clifford algebra $\mathbb{R}_m$ is then defined as the 2^m-dimensional real associative, but non-commutative, algebra generated by $\{e_1, \ldots, e_m\}$ and the multiplication rule: $e_j e_j + e_j e_i = -2\delta_{ij}$. An element of $\mathbb{R}_m$ is called a Clifford number and has the form $a = \sum_{A \subset M} a_A e_A$, $a_A \in \mathbb{R}$, where $A = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$ is a subset of $M = \{1, \ldots, m\}$ and $e_A = e_{i_1} \cdots e_{i_k}$. For A the empty set we put $e_\phi = 1$ and if A has k elements, e_A is called a k-vector. Denoting the projection of $a \in \mathbb{R}_m$ on its k-vector part as $[a]_k$, we get $a = \sum_{k=0}^m [a]_k$ with $[a]_k \in \mathbb{R}_m^{(k)}$. The *even subalgebra* is the subspace $\mathbb{R}_m^{(+)} = \sum_{k \text{ even}} \mathbb{R}_m^{(k)}$ of $\mathbb{R}^m$.

Vectors in $\mathbb{R}^m$ are identified with 1-vectors in $\mathbb{R}_m$. Note that the for two vectors $\underline{x}, \underline{y} \in \mathbb{R}_m$, the Clifford product $\underline{x}\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$ incorporates both the *inner product* $\underline{x} \cdot \underline{y} = -< \underline{x}, \underline{y} >$ and the *outer product* $\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i)$.

The conjugation on $\mathbb{R}_m$ is defined as the map sending $a \mapsto \bar{a}$, with $\overline{e_i} = -e_i$ and $\overline{ab} = \overline{b}\overline{a}$. The Clifford group $\Gamma(m)$ is the subgroup of $\mathbb{R}_m$ generated by the non-zero vectors of $\mathbb{R}^m$; the Pin group $\mathrm{Pin}(m)$ is the subgroup of $\Gamma(m)$ consisting of products of unit vectors of $\mathbb{R}^m$ and the Spin group $\mathrm{Spin}(m)$ is the subgroup of $\mathrm{Pin}(m)$ consisting of products of an *even* number of unit vectors of $\mathbb{R}^m$. For an element $s \in \mathrm{Pin}(m)$ the map $\chi(s) : \mathbb{R}^m \mapsto \mathbb{R}^m : \underline{x} \mapsto s\underline{x}\bar{s}$ induces an orthogonal transformation on $\mathbb{R}^m$. In this way $\mathrm{Pin}(m)$ defines a double covering of the orthogonal group $\mathrm{O}(m)$ whereas the Spin group defines a double covering of $\mathrm{SO}(m)$.

The Dirac operator on $\mathbb{R}^m$ is defined as the vector derivative $\underline{\partial} = \sum_j e_j \partial_j$, which is a first-order $\mathrm{Spin}(m)$-invariant differential operator factorizing the Laplacian Δ_m on $\mathbb{R}^m$: $\underline{\partial}^2 = -\Delta_m$. Let Ω be an open subset of $\mathbb{R}^m$ and let $f : \Omega \mapsto \mathbb{R}_m$ be an element of $C^1(\Omega)$. If $\underline{\partial} f = 0$ on Ω, f is called *monogenic* on Ω. It is clear that monogenic functions in Ω form a subclass of the harmonic functions in Ω. In polar coordinates the Dirac operator admits the following decomposition: $\underline{\partial} = \underline{\omega}(\partial_r + \frac{1}{r}\Gamma)$, where $\underline{x} = r\underline{\omega}$ and $\Gamma = -\underline{x} \wedge \underline{\partial}$ is the spherical Dirac operator on S^{m-1}. This operator, which is strongly related to the Atiyah-Singer Dirac operator on the sphere (see e.g. [17,20]) is a self-adjoint operator on the module $L_2(S^{m-1})$ with inner product

$$(f, g) = \int_{S^{m-1}} \bar{f} g \, \mathrm{d}\underline{\xi},$$

which is a consequence of Stokes' theorem on the sphere:

$$\int_{S^{m-1}} [(f\Gamma)g + f(\Gamma g)] \, \mathrm{d}\underline{\xi} = 0 = \int_{S^{m-1}} [(\overline{\Gamma f})g - \bar{f}(\Gamma g)] \, \mathrm{d}\underline{\xi}. \tag{1}$$

The restriction to S^{m-1} of a k-homogeneous monogenic polynomial $P_k(\underline{x})$ is called an *inner spherical* monogenic and satisfies $\Gamma P_k = -kP_k$. The restriction to S^{m-1} of a $(1 - k - m)$-homogeneous monogenic function $Q_k(\underline{x})$ on $\mathbb{R}^m \setminus \{\underline{0}\}$ is called an *outer spherical* monogenic and satisfies $\Gamma Q_k = (k + m - 1)Q_k$. The (right Clifford) modules containing these functions are, respectively, denoted as $M_+(k)$ and $M_-(k)$. Inner and outer spherical monogenics on S^{m-1} are related as follows: $P_k(\underline{\omega}) \in M_+(k) \Rightarrow \underline{\omega} P_k(\underline{\omega}) \in M_-(k)$ and vice versa. Each function $f \in L_2(S^{m-1})$ can be decomposed as $f = \sum_{k=0}^{\infty} P(k)[f] + Q(k)[f]$, where the series converges in L_2-sense (see Ref. [7]).

The fundamental solution for the Dirac operator is given by the so-called Cauchy kernel $E(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^m}$, satisfying $\underline{\partial} E(\underline{x}) = -\delta(\underline{x}) = E(\underline{x})\underline{\partial}$. Since the Dirac operator on $\mathbb{R}^m$ is invariant under translations, we immediately have that $\underline{\partial} E(\underline{x} - \underline{y}) = -\delta(\underline{x} - \underline{y})$.

2.2. Clifford analysis on hyperbolic space

The Clifford algebra $\mathbb{R}_{1,m}$ is generated by an orthonormal basis $\{\epsilon, e_1, \ldots, e_m\}$ for $\mathbb{R}^{1,m}$ and the multiplication rules $\epsilon e_j + e_j \epsilon = 0$, $e_i e_j + e_j e_i = -2\delta_{ij}$ and $\epsilon^2 = 1$. The Dirac operator on $\mathbb{R}^{1,m}$ is given by $\epsilon\partial_T - \underline{\partial}$, where $\underline{\partial} = \sum_j e_j \partial_{X_j}$. It was already mentioned in Section 1 that hyperbolic monogenics are defined as α-homogeneous solutions for the hyperbolic Dirac operator. Putting $F(T, \underline{X}) = \lambda^\alpha f(\underline{x})$, where $\lambda = T$ and $\underline{x} = \frac{\underline{X}}{T}$, and writing the Dirac operator on $\mathbb{R}^{1,m}$ in terms of $(\lambda, \underline{x})$ it is clear that the hyperbolic function theory is equivalent with a function theory for the operator $(\underline{\partial} + \epsilon[\mathbb{E} - \alpha])$ acting on functions $f(\underline{x})$ defined on the unit ball $B_m(1) \subset \mathbb{R}^m$ (for more details we refer to [8–10]). The following definition and theorem yield a method to construct hyperbolic monogenics.

Definition 1. For all $\underline{x} \in B_m(1)$ the function $\mathrm{Mod}(\alpha, k, \underline{x})$ is defined as

$$\mathrm{Mod}(\alpha, k, \underline{x}) = F_1(|\underline{x}|^2) + \frac{k-\alpha}{2k+m}\underline{x}\epsilon F_2(|\underline{x}|^2),$$

where

$$F_1(t) = F\left(\frac{1+k-\alpha}{2}, \frac{k-\alpha}{2}, k+\frac{m}{2}; t\right),$$

$$F_2(t) = F\left(\frac{1+k-\alpha}{2}, \frac{2+k-\alpha}{2}, 1+k+\frac{m}{2}; t\right).$$

Theorem 1. *Let $P_k(\underline{\omega}) \in M_+(k)$ be an inner spherical monogenic and let $\alpha \in \mathbb{C}$. Then the function $P_{\alpha,k}(T, \underline{X})$ given for all $(T, \underline{X}) \in FC$ by*

$$P_{\alpha,k}(T, \underline{X}) = T^{\alpha}\mathrm{Mod}\left(\alpha, k, \frac{\underline{X}}{T}\right) P_k\left(\frac{\underline{X}}{T}\right)$$

is an α-homogeneous solution for the hyperbolic Dirac operator.

Note that the value $\alpha = -m/2$ corresponds to the conformal Dirac operator on the hyperbolic unit ball (see e.g. [17]). The conformal Dirac operator is invariant under the Moebius group $\mathrm{Mob}(m) \cong \mathrm{Spin}(2, m)$, and by restricting this group to the subgroup $\mathrm{Spin}(1, m)$, which is the invariance group of the Dirac operator on the hyperbolic unit ball, a richer function theory is obtained.

3. Distributions defined by divergent integrals

In this section we introduce the distributions x_+^λ on $\mathcal{D}(\mathbb{R})$ and the distributions ρ^λ on $\mathcal{D}(\mathbb{R}^{1,m})$, with λ an arbitrary complex number. As a general reference to the rest of this section, we refer to [6,12,14,16].

The function x_+^λ, defined by x^λ for $x > 0$ and $x = 0$ elsewhere, is locally integrable for $\mathrm{Re}(\lambda) > -1$ and hence defines a regular distribution for these values. To define x_+^λ in the strip $-n-1 < \mathrm{Re}(\lambda) < -n$, one can use analytic continuation:

$$< x_+^\lambda, \varphi >= \frac{< (\mathrm{d}^n/\mathrm{d}x^n)x_+^{\lambda+n}, \varphi >}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)},$$

where the derivatives with respect to x must be interpreted in distributional sense. Hence, for $-n-1 < \mathrm{Re}(\lambda) < -n$ one defines

$$< x_+^\lambda, \varphi >= (-1)^n \frac{< x_+^{\lambda+n}, \varphi^{(n)} >}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

This means that for all $\varphi \in \mathcal{D}(\mathbb{R})$, $< x_+^\lambda, \varphi >$ defines a meromorphic function of λ with simple poles at $\lambda = -1-n$, $n \in \mathbb{N}$. The residue at $\lambda = -1-n$ is easily found to be $\mathrm{Res}(x_+^\lambda, \lambda = -1-n) = \frac{(-1)^n}{n!}\delta^{(n)}(x)$, for all $n \in \mathbb{N}$. In order to remove the simple poles of x_+^λ we divide by $\Gamma(1+\lambda)$, and so the distribution $\frac{x_+^\lambda}{\Gamma(\lambda+1)}$ is well-defined on $\mathcal{D}(\mathbb{R})$ for all $\lambda \in \mathbb{C}$ with $< \frac{x_+^\lambda}{\Gamma(\lambda+1)}, \varphi >$ a holomorphic function of λ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

In what follows, we will encounter the Beta-type integral $I_B(\lambda, \mu)$ defined for arbitrary complex λ and μ as $I_B(\lambda, \mu) = \int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\,\mathrm{d}t$. For $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\mu) > 0$ this integral converges in the classical sense to the Beta function $B(\lambda, \mu)$, defined in terms of the Gamma function as

$$B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

For more general λ and μ this relation remains valid, and this can easily be seen as follows: the Beta integral $I_B(\lambda, \mu)$ can actually be interpreted as the distribution $t_+^{\lambda-1}(1-t)_+^{\mu-1}$ acting on the constant function 1. The product of the distributions $t_+^{\lambda-1}$ and $(1-t)_+^{\mu-1}$ is well-defined for those values for which they, respectively, exist (i.e. for $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\mu) > 0$), as they have a "problematic behaviour" for different values (in case 0 and 1), and it has compact support $[0, 1] =]-\infty, 1] \cap [0, +\infty[$. This means that as a function of (λ, μ), this product is defined in $\mathbb{C}^2 \setminus \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda \in -\mathbb{N}, \mu \in -\mathbb{N}\}$ which is the complex plane minus a grid. This distribution yields the Beta function $B(\lambda, \mu)$ in the complex strip $\{(\lambda, \mu) \in \mathbb{C}^2 : \mathrm{Re}(\lambda) > 0, \mathrm{Re}(\mu) > 0\}$ and for all other possible values this equality follows by analytic continuation. We may thus conclude that for all $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(\lambda, \mu) \in \mathbb{C}^2 : \lambda \in -\mathbb{N}, \mu \in -\mathbb{N}\}$ we have:

$$I_B(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}. \tag{2}$$

Next we consider the distribution ρ^λ, with $\lambda \in \mathbb{C}$. The function ρ is hereby defined for space–time vectors $(T, \underline{X}) \in \mathbb{R}^{1,m}$ by $(T^2 - |\underline{X}|)^{1/2}$ if $(T, \underline{X}) \in FC$ and $\rho = 0$ elsewhere. In the half-plane $\mathrm{Re}(\lambda) > -2$, the function ρ^λ defines a regular distribution since ρ^λ is locally integrable for these values of λ. Indeed,

$$< \rho^\lambda, \varphi >= \int\int (T^2 - |\underline{X}|^2)^{\lambda/2}(T, \underline{X})\varphi(T, \underline{X})\,\mathrm{d}T\,\mathrm{d}\underline{X}$$

defines an analytic function when $\mathrm{Re}(\lambda) > -2$ for $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$. Analytic continuation can be used to extend $< \rho^\lambda, \varphi >$ to a meromorphic function in the whole complex plane.

For that purpose we introduce the wave-operator $\Box = \partial_T^2 - \Delta_m$ on $\mathbb{R}^{1,m}$. The fact that $\Box\rho^\lambda = \lambda(\lambda + m - 1)\rho^{\lambda-2}$, suggests the following definition for ρ^λ in the strip $-2n - 2 < \lambda < -2n$:

$$< \rho^\lambda, \varphi >= \frac{< \Box^n \rho^{\lambda+2n}, \varphi >}{(\lambda + 2)(\lambda + 4)\cdots(\lambda + 2n)(\lambda + m + 1)\cdots(\lambda + m + 2n - 1)}.$$

Hence, ρ^λ has poles at $\lambda = -2 - 2n, n \in \mathbb{N}$ and at $\lambda = -1 - m - 2n, n \in \mathbb{N}$. For m even all the poles are simple, while for m odd the points $-2, -4, \ldots, 1 - m$ are simple poles and the points $-m - 1, -m - 3, \ldots$ are double poles.

The distributions ρ^λ are normalized by introducing suitable factors. Putting

$$Z_\mu = \frac{\rho^{\mu-m-1}}{\pi^{((m-1)/2)} 2^{\mu-1} \Gamma\left(\frac{\mu}{2}\right) \Gamma\left(\frac{\mu+1-m}{2}\right)}, \tag{3}$$

the functional $< Z_\mu, \varphi >$ becomes an entire function of the complex variable μ for each test function $\varphi \in \mathcal{D}(\mathbb{R}^{1,m})$. These so-called Riesz-distributions Z_μ enjoy remarkable properties, a few of which will be listed here:

(1) The support of Z_μ is contained in the set $\overline{FC}$.
(2) The distributions Z_μ satisfy $Z_\mu * Z_\nu = Z_{\mu+\nu}$.
(3) For all $k \in \mathbb{N}$, we have: $Z_{-2k} = \Box^k \delta(X)$, with $\delta(X) = \delta(T)\delta(\underline{X})$ the delta-function in space–time co-ordinates.
(4) For all $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$, $\Box^k Z_\mu = Z_{\mu-2k}$.

Let us then introduce $\mathcal{D}'_+(\mathbb{R}^{1,m})$ as the set of distributions $f \in \mathcal{D}'(\mathbb{R}^{1,m})$ with a support contained in $\overline{FC}$. Taking the convolution of two elements of $\mathcal{D}'_+(\mathbb{R}^{1,m})$, the result is again in $\mathcal{D}'_+(\mathbb{R}^{1,m})$ and hence $\mathcal{D}'_+(\mathbb{R}^{1,m})$ is a convolution algebra. The distributions Z_μ belong to $\mathcal{D}'_+(\mathbb{R}^{1,m})$, and their *unique* inverses in $\mathcal{D}'_+(\mathbb{R}^{1,m})$ are the distributions $Z_{-\mu}$: $Z_\mu * Z_{-\mu} = \delta(X)$, $\mu \in \mathbb{C}$. It follows that the differential equation $\Box^k f = g$, with f and g belonging to $\mathcal{D}'_+(\mathbb{R}^{1,m})$, has unique solution $f = Z_{2k} * g$.

4. The photogenic Cauchy kernel

In this section we solve the so-called photogenic Dirac equation for the hyperbolic fundamental solution having singularities *on* the nullcone. Note that it does *not* suffice to solve the equation $(\epsilon\partial_T - \underline{\partial})E(T, \underline{X}) = \delta(T\underline{\omega} - \underline{X})$, with $\underline{\omega} \in S^{m-1}$, although the right-hand side of this equation represents a delta function on the nullcone. In view of the projective nature of our model for the hyperbolic unit ball, we convolute the right-hand side with the distribution $T_+^{\alpha+m-1}$, which expresses the homogeneous character (see e.g. [8]). This leads to the photogenic Dirac equation:

$$(\epsilon\partial_T - \underline{\partial})E_{\alpha,\underline{\omega}}(T, \underline{X}) = T_+^{\alpha+m-1}\delta(\underline{X} - T\underline{\omega}),$$

where we have chosen to label the fundamental solution with both its degree of homogeneity and the arbitrary unit vector $\underline{\omega}$. Note that we expect the distribution $E_{\alpha,\underline{\omega}}(T, \underline{X})$ to be undefined at the values $\alpha \in -m - \mathbb{N}$, because the right-hand side is not defined for these values. To solve this equation, we use the fact that $(\epsilon\partial_T - \underline{\partial})^2 = \Box$. Hence, we first consider the scalar problem $\Box\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) = T_+^{\alpha+m-1}\delta(\underline{X} - T\underline{\omega})$. As the right-hand side is contained in the set $\mathcal{D}'_+(\mathbb{R}^{1,m})$, we get: $\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) = Z_2 * T_+^{\alpha+m-1}\delta(T\underline{\omega} - \underline{X})$. Using the definition for the Riesz-distribution Z_2, this leads to:

$$\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) = \frac{1}{2\pi^{m-1/2}\Gamma(3 - m/2)} \int_0^{S_0} \frac{S^{\alpha+m-1}}{[T^2 - |\underline{X}|^2 - 2S(T- < \underline{X}, \underline{\omega} >)]^{m-1/2}} \mathrm{d}S,$$

where we have put $S_0 = \frac{T^2-|\underline{X}|^2}{2(T-<\underline{X},\underline{\omega}>)}$. Recalling the definition of the Beta integral, we get *for even dimensions m*:

$$E_{\alpha,\underline{\omega}}(T, \underline{X}) = (\epsilon\partial_T - \underline{\partial})\left[\frac{\Gamma(\alpha + m)}{2^{1+\alpha+m}\pi^{m-1/2}\Gamma(\alpha + (m + 3/2)} \frac{(T^2 - |\underline{X}|^2)^{\alpha+((m+1)/2)}}{(T- < \underline{X}, \underline{\omega} >)^{\alpha+m}}\right].$$

Denoting the constant in previous expression as $c(\alpha, m)$, we are lead to:

$$\begin{aligned} E_{\alpha,\underline{\omega}}(T, \underline{X}) = {} & (2\alpha + m + 1)c(\alpha, m)(\epsilon T + \underline{X})\frac{(T^2 - |\underline{X}|^2)^{\alpha+((m-1)/2)}}{(T- < \underline{X}, \underline{\omega} >)^{\alpha+m}} \\ & - (\alpha + m)c(\alpha, m)(\epsilon + \underline{\omega})\frac{(T^2 - |\underline{X}|^2)^{\alpha+((m+1)/2)}}{(T- < \underline{X}, \underline{\omega} >)^{\alpha+m+1}}. \end{aligned}$$

This expression for $E_{\alpha,\underline{\omega}}(T, \underline{X})$ is however valid for *both* even and odd m. Although the Gamma function in the denominator of the expression for $c(\alpha, m)$ seems to remove the poles at $\alpha \in -m - \mathbb{N}$ (coming from the Gamma function in the nominator) in case of an odd dimension, there *are* poles at these values for α for *both* even and odd dimensions m. To illustrate this, we consider the residue for $\alpha = -m$. In that case, we have:

$$\mathrm{Res}\{T_+^{\alpha+m-1}\delta(T\underline{\omega} - \underline{X}), \alpha = -m\} = \delta(T)\delta(T\underline{\omega} - \underline{X}) = \delta(T)\delta(\underline{X}).$$

This means that $\mathrm{Res}\{Z_2 * T_+^{\alpha+m-1}\delta(T\underline{\omega} - \underline{X}), \alpha = -m\} = Z_2$. On the other hand, we also have by definition:

$$\mathrm{Res}\{\Phi_{\alpha,\underline{\omega}}(T, \underline{X}), \alpha = -m\} = \lim_{\alpha\to -m}(\alpha + m)\Phi_{\alpha,\underline{\omega}}(T, \underline{X}) = Z_2.$$

Thus, although the simplification

$$\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+(m+3/2))} = (\alpha+m-1)\cdots\left(\alpha+\frac{m+3}{2}\right)$$

seems to remove the pole at $\alpha=-m$, it only makes it less obvious to see that there actually *is* a pole at this value. Because we now have two distributions which are equal in a strip of the complex plane and which have poles at the same values for α, they are equal in the whole complex plane by analytic continuation.

5. The photogenic Cauchy transform

Now that we have found a photogenic Cauchy kernel we define a photogenic Cauchy transform. To do so, we will use the equivalence between the function theory for the Dirac operator on the hyperbolic unit ball and the function theory for the operator $(\underline{\partial}+\epsilon[\mathbb{E}-\alpha])$ on $B_m(1)$. In terms of the geometrical model, this means that we project the photogenic Dirac equation onto the hyperplane $\Pi \leftrightarrow T=1 \subset \mathbb{R}^{1,m}$. Reintroducing $(\lambda,\underline{x})$ as coordinates in the FC, we get immediately that $(\underline{\partial}+\epsilon[\mathbb{E}-\alpha])E_\alpha(\underline{x},\underline{\omega}) = -\delta(\underline{x}-\underline{\omega})$. The fundamental solution $E_\alpha(\underline{x},\underline{\omega})$ is given by

$$\begin{aligned} E_\alpha(\underline{x},\underline{\omega}) &= (2\alpha+m+1)c(\alpha,m)(\epsilon+\underline{x})\frac{(1-r^2)^{\alpha+((m-1)/2)}}{(1-<\underline{x},\underline{\omega}>)^{\alpha+m}} \\ &\quad -(\alpha+m)c(\alpha,m)(\epsilon+\underline{\omega})\frac{(1-r^2)^{\alpha+((m+1)/2)}}{(1-<\underline{x},\underline{\omega}>)^{\alpha+m+1}}, \end{aligned}$$

where we have put $\underline{x}=r\underline{\xi}$. We then define the photogenic Cauchy transform of a function $f(\underline{\omega})$ on S^{m-1}, for all $\underline{x}\in B(1)$, as

$$\mathcal{C}_F^\alpha[f](\underline{x}) = \frac{1}{A_m}\int_{S^{m-1}} E_\alpha(\underline{x},\underline{\omega})\underline{\omega} f(\underline{\omega})\,\mathrm{d}\underline{\omega},$$

where the additional factor $\underline{\omega}$ plays the role of unit normal vector on S^{m-1}. The result is a function $\mathcal{C}_F^\alpha[f](\underline{x})$, defined on $B(1)$, which is a solution for the operator $(\underline{\partial}+\epsilon(\mathbb{E}-\alpha))$. In view of the projective nature of our model for the hyperbolic unit ball, this means that $\mathcal{C}_F^\alpha[f](T,\underline{X}) = T^\alpha \mathcal{C}_F^\alpha[f]\left(\frac{\underline{X}}{T}\right)$ is a hyperbolic monogenic function for all $(T,\underline{X})\in FC$. Because functions $f\in L_2(S^{m-1})$ can be decomposed in a series of inner and outer spherical monogenics, we now calculate two things:

(1) for arbitrary $P_k(\underline{\omega})\in M_+(k)$ we determine $\mathcal{C}_F^\alpha[P_k](\underline{x})$;
(2) for arbitrary $Q_k(\underline{\omega})\in M_-(k)$ we determine $\mathcal{C}_F^\alpha[Q_k](\underline{x})$.

We will hereby make use of two lemmata. The first one is a refinement of the classical Hecke–Funk theorem (see e.g. [15]), and the latter yields a formula for an integral that will often occur in what follows.

Lemma 1. see [19]

Let $P_k(\underline{\omega}) \in M_+(k)$ and denote the Legendre polynomial of degree k in m dimensions by $P_{k,m}(t)$. Putting $\underline{x} = r\underline{\xi}$, we get:

$$\int_{S^{m-1}} f(<\underline{x},\underline{\omega}>)P_k(\underline{\omega})\,\mathrm{d}\underline{\omega} = A_m\left(\int_{-1}^{1} f(rt)P_{k,m}(t)(1-t^2)^{(m-3)/2}\,\mathrm{d}t\right)P_k(\underline{\xi}),$$

$$\int_{S^{m-1}} f(<\underline{x},\underline{\omega}>)\underline{\omega}P_k(\underline{\omega})\,\mathrm{d}\underline{\omega}$$

$$= A_m\left(\int_{-1}^{1} f(rt)P_{1+k,m}(t)(1-t^2)^{(m-3)/2}\,\mathrm{d}t\right)\underline{\xi}P_k(\underline{\xi}).$$

Lemma 2. *Let $F(a, b; c; t)$ be Gauss' hypergeometric series and let $r < 1$. We then have the following identity*

$$\int_{-1}^{1} \frac{P_{k,m}(t)(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t$$

$$= \frac{\pi^{(1/2)}(\lambda)_k\Gamma((m-1)/2)}{2^k(m/2)_k\Gamma(m/2)}r^kF\left(\frac{k+\lambda}{2},\frac{1+k+\lambda}{2};k+\frac{m}{2};r^2\right),$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochammer symbol, with $(a)_0 = 1$.

Proof. This lemma can be proved by induction on the parameter k. For $k = 0$ and $r < 1$:

$$\int_{-1}^{1} \frac{(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t = \sum_{l=0}^{\infty}\binom{-\lambda}{2l}(r)^{2l}\int_0^1 (t^2)^{l-(1/2)}(1-t^2)^{(m-3)/2}\mathrm{d}t^2,$$

which by means of the definition of the Beta function and properties of the Gamma function, reduces to

$$\int_{-1}^{1} \frac{(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t = \sqrt{\pi}\frac{\Gamma((m-1)/2)}{\Gamma(m/2)}F\left(\frac{\lambda}{2},\frac{1+\lambda}{2};\frac{m}{2};r^2\right).$$

For $k = 1$ we get, with $P_{1,m}(t) = t$:

$$\int_{-1}^{1} \frac{t(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t = \frac{\partial_r}{\lambda-1}\int_{-1}^{1} \frac{(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda-1}}\mathrm{d}t,$$

which, by means of the fact that $\frac{\mathrm{d}}{\mathrm{d}x}F(a,b;c;x) = \frac{ab}{c}F(1+a,1+b;1+c;x)$, reduces to

$$\int_{-1}^{1} \frac{t(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t = \sqrt{\pi}\frac{\lambda\Gamma((m-1)/2)}{m\Gamma(m/2)}rF\left(\frac{1+\lambda}{2},1+\frac{\lambda}{2};1+\frac{m}{2};r^2\right).$$

The rest of the proof makes use of the recurrence formula for the Legendre polynomials (see e.g. [15]):

$$\int_{-1}^{1} \frac{P_{1+k,m}(t)(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t = \frac{2k+m-2}{k+m-2}\frac{\partial_r}{\lambda-1}\int_{-1}^{1}\frac{P_{k,m}(t)(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda-1}}\mathrm{d}t$$
$$\times - \frac{k}{k+m-2}\int_{-1}^{1}\frac{P_{k-1,m}(t)(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda-1}}\mathrm{d}t.$$

Using elementary properties of the hypergeometric series and the induction hypothesis, this can be simplified and yields

$$\frac{\pi^{(1/2)}(\lambda)_{1+k}\Gamma((m-1)/2)}{2^{1+k}(m/2)_{1+k}\Gamma(m/2)}r^{1+k}F\left(\frac{1+k+\lambda}{2}, 1+\frac{k+\lambda}{2}; 1+k+\frac{m}{2}; r^2\right).$$

This proves the Lemma. □

5.1. The photogenic Cauchy transform of inner spherical monogenics

Consider an arbitrary element $P_k(\underline{\omega}) \in M_+(k)$. By definition, we have:

$$\mathcal{C}_F^{\alpha}[P_k](\underline{x}) = \frac{1}{A_m}\int_{S^{m-1}} E_{\alpha}(\underline{x}, \underline{\omega})\underline{\omega}P_k(\underline{\omega})\,\mathrm{d}\underline{\omega}.$$

The photogenic Cauchy transform has a bivector component $\mathcal{C}_F^{\alpha}[P_k]_2$ and a scalar component $\mathcal{C}_F^{\alpha}[P_k]_0$. Introducing the short-hand notation $\mathcal{P}(k, \lambda, r)$ for

$$\mathcal{P}(k, \lambda; r) = \int_{-1}^{1}\frac{P_{k,m}(t)(1-t^2)^{(m-3)/2}}{(1-rt)^{\lambda}}\mathrm{d}t,$$

we get:

$$\mathcal{C}_F^{\alpha}[P_k]_2 = c(\alpha, m)\begin{bmatrix} (2\alpha+m+1)(1-r^2)^{\alpha+((m-1)/2)}\mathcal{P}(1+k, \alpha+m; r) \\ -(\alpha+m)(1-r^2)^{\alpha+((m+1)/2)}\mathcal{P}(1+k, \alpha+m+1; r)\end{bmatrix}\epsilon\underline{\xi}P_k(\underline{\xi}),$$
$$\mathcal{C}_F^{\alpha}[P_k]_0 = -c(\alpha, m)\begin{bmatrix} (2\alpha+m+1)(1-r^2)^{\alpha+((m-1)/2)}r\mathcal{P}(1+k, \alpha+m; r) \\ -(\alpha+m)(1-r^2)^{\alpha+((m+1)/2)}\mathcal{P}(k, \alpha+m+1; r)\end{bmatrix}P_k(\underline{\xi}).$$

With the aid of Lemma 2, this reduces to

$$\mathcal{C}_F^{\alpha}[P_k]_2 = c(\alpha, m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k+1)\Gamma((m-1)/2)}{2^{k+1}\Gamma(1+k+(m/2)\Gamma(\alpha+m)}$$
$$\times (1-r^2)^{\alpha+((m-1)/2)}\epsilon\underline{x}P_k(\underline{x})$$

$$\times\left[\begin{array}{l}(2\alpha+m+1)F\left(\frac{1+\alpha+m+k}{2},1+\frac{\alpha+m+k}{2};1+k+\frac{m}{2};r^2\right)\\ -(\alpha+m+k+1)(1-r^2)F\\ \times\left(1+\frac{\alpha+m+k}{2},1+\frac{1+\alpha+m+k}{2};1+k+\frac{m}{2};r^2\right)\end{array}\right],$$

$$\mathcal{C}_F^{\alpha}[P_k]_0=-c(\alpha,m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k+1)\Gamma((m-1)/2)}{2^k\Gamma(k+(m/2))\Gamma(\alpha+m)}(1-r^2)^{\alpha+((m-1)/2)}P_k(\underline{x})$$
$$\times\left[\begin{array}{l}\frac{2\alpha+m+1}{2k+m}r^2F\left(\frac{1+\alpha+m+k}{2},1+\frac{\alpha+m+k}{2};1+k+\frac{m}{2};r^2\right)\\ -(1-r^2)F\left(\frac{1+\alpha+m+k}{2},1+\frac{\alpha+m+k}{2};k+\frac{m}{2};r^2\right)\end{array}\right].$$

Eventually making use of the definition of the hypergeometric series to simplify the terms between square brackets and using Kummer's relation, we find:

$$\begin{aligned}\mathcal{C}_F^{\alpha}[P_k](\underline{x})&=\mathcal{C}_F^{\alpha}[P_k]_0+\mathcal{C}_F^{\alpha}[P_k]_2\\ &=c(\alpha,m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k+1)\Gamma((m-1)/2)}{2^k\Gamma(k+(m/2))\Gamma(\alpha+m)}\mathrm{Mod}(\alpha,k,\underline{x})P_k(\underline{x})\end{aligned}$$

with $\mathrm{Mod}(\alpha, k, \underline{x})$ given by Definition 1.

5.2. The photogenic Cauchy transform of outer spherical monogenics

Next, we consider an arbitrary outer spherical monogenic $Q_k(\underline{\omega})\in M_-(k)$. Note that we restrict ourselves to $Q_k(\underline{\omega})=\underline{\omega}P_k(\underline{\omega})$ such that $P_k(\underline{\omega})\in M_+(k)$ takes its values in the even subalgebra, whence $[P_k(\underline{\omega}),\epsilon]=0$. We have by definition:

$$\mathcal{C}_F^{\alpha}[Q_k](\underline{x})=-\frac{1}{A_m}\int_{S^{m-1}}E_\alpha(\underline{x},\underline{\omega})P_k(\underline{\omega})\,\mathrm{d}\underline{\omega}.$$

This time the photogenic Cauchy transform consists, up to the term $P_k(\underline{\xi})$, of a part $\mathcal{C}_F^{\alpha}[Q_k]_\epsilon$ in ϵ and a part $\mathcal{C}_F^{\alpha}[Q_k]_{\underline{\xi}}$ in $\underline{\xi}$, respectively, given by (hereby making use of Lemma 1):

$$\mathcal{C}_F^{\alpha}[Q_k]_\epsilon=c(\alpha,m)\left[\begin{array}{l}-(2\alpha+m+1)(1-r^2)^{\alpha+((m-1)/2)}\mathcal{P}(k,\alpha+m;r)\\ +(\alpha+m)(1-r^2)^{\alpha+((m+1)/2)}\mathcal{P}(k,\alpha+m+1;r)\end{array}\right]\epsilon P_k(\underline{\xi}),$$

$$\mathcal{C}_F^{\alpha}[Q_k]_{\underline{\xi}}=-c(\alpha,m)\left[\begin{array}{l}(2\alpha+m+1)(1-r^2)^{\alpha+((m-1)/2)}r\mathcal{P}(k,\alpha+m;r)\\ -(\alpha+m)(1-r^2)^{\alpha+((m+1)/2)}\mathcal{P}(1+k,\alpha+m+1;r)\end{array}\right]$$
$$\times\underline{\xi}P_k(\underline{\xi}).$$

With the aid of Lemma 2, this reduces to

$$\mathcal{C}_F^\alpha[P_k]_\epsilon = c(\alpha, m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k)\Gamma((m-1)/2)}{2^k\Gamma(k+(m/2))\Gamma(\alpha+m)}(1-r^2)^{\alpha+((m-1)/2)}\epsilon P_k(\underline{x})$$
$$\times\left[\begin{array}{l} -(2\alpha+m+1)F\left(\dfrac{\alpha+m+k}{2}, \dfrac{1+\alpha+m+k}{2}; k+\dfrac{m}{2}; r^2\right) \\ +(\alpha+m+k)(1-r^2)F \\ \times\left(1+\dfrac{\alpha+m+k}{2}, \dfrac{1+\alpha+m+k}{2}; k+\dfrac{m}{2}; r^2\right)\end{array}\right],$$

$$\mathcal{C}_F^\alpha[P_k]_{\underline{\xi}} = -c(\alpha, m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k+1)\Gamma((m-1)/2)}{2^k\Gamma(k+(m/2))\Gamma(\alpha+m)}$$
$$\times(1-r^2)^{\alpha+((m-1)/2)}\underline{x}P_k(\underline{x})$$
$$\times\left[\begin{array}{l} \dfrac{2\alpha+m+1}{\alpha+m+k}F\left(\dfrac{\alpha+m+k}{2}, \dfrac{1+\alpha+m+k}{2}; k+\dfrac{m}{2}; r^2\right) \\ -(1-r^2)\dfrac{\alpha+m+k+1}{2k+m}F \\ \times\left(1+\dfrac{\alpha+m+k}{2}, 1+\dfrac{1+\alpha+m+k}{2}; 1+k+\dfrac{m}{2}; r^2\right)\end{array}\right].$$

Eventually making use of the definition of the hypergeometric series to simplify the terms between square brackets we get:

$$\mathcal{C}_F^\alpha[Q_k](\underline{x}) = \mathcal{C}_F^\alpha[Q_k]_\epsilon + \mathcal{C}_F^\alpha[Q_k]_{\underline{\xi}}$$
$$= -(1+\alpha-k)c(\alpha, m)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k)\Gamma((m-1)/2)}{2^k\Gamma(k+(m/2))\Gamma(\alpha+m)}$$
$$\times\,\mathrm{Mod}(\alpha, k, \underline{x})P_k(\underline{x})\epsilon.$$

6. Photogenic boundary values

Now that we have found the photogenic Cauchy transforms of inner and outer spherical monogenic functions on $\mathbb{R}^m$, we will determine their boundary values. By that, we mean the following: as both $\mathcal{C}_F^\alpha[P_k](\underline{x})$ and $\mathcal{C}_F^\alpha[Q_k](\underline{x})$ are solutions to the operator $(\underline{\partial}+\epsilon[\mathbb{E}-\alpha])$ defined in the unit ball $B_m(1) \subset \mathbb{R}^m$, one can wonder whether the following limits exist:

$$\mathcal{C}_F^\alpha[P_k]\uparrow(\underline{\xi}) = \lim_{r\to 1^-}[H(1-r)\mathcal{C}_F^\alpha[P_k](r\underline{\xi})],$$
$$\mathcal{C}_F^\alpha[Q_k]\uparrow(\underline{\xi}) = \lim_{r\to 1^-}[H(1-r)\mathcal{C}_F^\alpha[Q_k](r\underline{\xi})].$$

In order to calculate these limits, we will make use of the following property of the hypergeometric function: for $\mathrm{Re}(c-a-b) > 0$ and $c \notin -\mathbb{N}$, we have

$$\lim_{t\to 1} F(a,b;c;t) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}. \tag{4}$$

Recalling the definitions of the function $\mathrm{Mod}(\alpha, k, \underline{x})$ and the constant $c(\alpha, m)$, one can easily verify that for $\mathrm{Re}(\alpha) + \frac{m-1}{2} > 0$

$$\mathcal{C}_F^\alpha[P_k] \uparrow (\underline{\xi}) = \frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{(\alpha+m+k)\{(\alpha+m+k-1)+(k-\alpha)\underline{\xi}\epsilon\}P_k(\underline{\xi})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}, \tag{5}$$

$$\mathcal{C}_F^\alpha[Q_k] \uparrow (\underline{\xi}) = \frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{(1+\alpha-k)\{(\alpha-k)-(\alpha+m+k-1)\underline{\xi}\epsilon\}Q_k(\underline{\xi})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}. \tag{6}$$

Both formulae can be represented by means of the spherical angular operator $\Gamma_{\underline{\xi}}$ on S^{m-1} as follows:

$$\mathcal{C}_F^\alpha[P_k] \uparrow (\underline{\xi}) = \frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{\{(\alpha+m-1-\Gamma_{\underline{\xi}})-\underline{\xi}\epsilon(\Gamma_{\underline{\xi}}+\alpha)\}(\alpha+m-\Gamma_{\underline{\xi}})P_k(\underline{\xi})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))},$$

$$\mathcal{C}_F^\alpha[Q_k] \uparrow (\underline{\xi}) = \frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{\{(\alpha+m-1-\Gamma_{\underline{\xi}})-\underline{\xi}\epsilon(\Gamma_{\underline{\xi}}+\alpha)\}(\alpha+m-\Gamma_{\underline{\xi}})Q_k(\underline{\xi})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}.$$

This shows that for $\mathrm{Re}(\alpha) + \frac{m-1}{2} > 0$ the boundary values $\mathcal{C}_F^\alpha[P_k] \uparrow (\underline{\xi})$ and $\mathcal{C}_F^\alpha[Q_k] \uparrow (\underline{\xi})$ can be found by letting the polynomial operator

$$\mathcal{P}_\alpha(\Gamma_{\underline{\xi}}) = \frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{\{(\alpha+m-1-\Gamma_{\underline{\xi}})-\underline{\xi}\epsilon(\Gamma_{\underline{\xi}}+\alpha)\}(\alpha+m-\Gamma_{\underline{\xi}})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}$$

in $\Gamma_{\underline{\xi}}$ act, respectively, on the spherical monogenics $P_k(\underline{\xi})$ and $Q_k(\underline{\xi})$. These formulae can be reinterpreted, in such a way that they can be recovered as the action of a distribution on the spherical monogenics. Two different approaches can be followed. The first one uses the above facts:

$$\mathcal{C}_F^\alpha[P_k] \uparrow (\underline{\xi}) = \int_{S^{m-1}} \delta(\underline{\xi}-\underline{\omega})\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})P_k(\underline{\omega})\,\mathrm{d}\underline{\omega},$$

$$\mathcal{C}_F^\alpha[Q_k] \uparrow (\underline{\xi}) = \int_{S^{m-1}} \delta(\underline{\xi}-\underline{\omega})\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})Q_k(\underline{\omega})\,\mathrm{d}\underline{\omega}.$$

With the aid of (1), this becomes:

$$\mathcal{C}_F^\alpha[P_k] \uparrow (\underline{\xi}) = \int_{S^{m-1}} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})\delta(\underline{\xi}-\underline{\omega})}P_k(\underline{\omega})\,\mathrm{d}\underline{\omega},$$

$$\mathcal{C}_F^\alpha[Q_k] \uparrow (\underline{\xi}) = \int_{S^{m-1}} \overline{\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})\delta(\underline{\xi}-\underline{\omega})}Q_k(\underline{\omega})\,\mathrm{d}\underline{\omega}.$$

If we now define the action of a distribution $\mathcal{D}(\underline{\omega})$ on a test function $\varphi(\underline{\omega})$ by

$$< \mathcal{D}(\underline{\omega}), \varphi(\underline{\omega}) >= \int_{S^{m-1}} \mathcal{D}(\underline{\omega})\underline{\omega}\varphi(\underline{\omega})\,\mathrm{d}\underline{\omega}, \tag{7}$$

we get for all complex α such that $\mathrm{Re}(\alpha) + \frac{m-1}{2} > 0$:

$$\mathcal{C}_F^{\alpha}[P_k] \uparrow (\underline{\xi}) =< \overline{\underline{\omega}\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})\delta(\underline{\xi}-\underline{\omega})}, P_k(\underline{\omega}) >,$$

$$\mathcal{C}_F^{\alpha}[Q_k] \uparrow (\underline{\xi}) =< \overline{\underline{\omega}\mathcal{P}_\alpha(\Gamma_{\underline{\omega}})\delta(\underline{\xi}-\underline{\omega})}, Q_k(\underline{\omega}) > .$$

On the other hand, we can also use the photogenic Cauchy kernel and interpret $E_\alpha(\underline{x}, \underline{\omega})$, with $\underline{x} = r\underline{\xi}$ as a distribution in $\underline{\xi}$. When acting on a test function $\varphi(\underline{\xi})$, the action then becomes (using the definition in expression (7)):

$$< E_\alpha(r\underline{\xi}, \underline{\omega}), \varphi(\underline{\xi}) >= \int_{S^{m-1}} E_\alpha(r\underline{\xi}, \underline{\omega})\underline{\xi}\varphi(\underline{\xi})\,\mathrm{d}\underline{\xi}.$$

If we let this distribution act upon $P_k(\underline{\xi})$ and $Q_k(\underline{\xi})$, respectively, we get (hereby using similar calculations as in Section 5):

$$\begin{aligned}
&< E_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) > \\
&\quad = c(\alpha, m)\frac{((k-\alpha)/2)}{k+(m/2)}\frac{\pi^{(1/2)}\Gamma(\alpha+m+k+1)\Gamma((m-1)/2)}{2^k\Gamma(\alpha+m)\Gamma(k+(m/2))} \\
&\qquad \times r^{1+k}\underline{\omega}\epsilon\, F\left(\frac{1+k-\alpha}{2}, 1+\frac{k-\alpha}{2}; 1+k+\frac{m}{2}; r^2\right) \\
&\qquad \times P_k(\underline{\omega}) + c(\alpha, m)(\alpha-k)(\alpha-k+1)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k)\Gamma((m-1)/2)}{2^{k+1}\Gamma(\alpha+m)\Gamma(1+k+(m/2)} \\
&\qquad \times r^k F\left(\frac{1+k-\alpha}{2}, 1+\frac{k-\alpha}{2}; 1+k+\frac{m}{2}; r^2\right) P_k(\underline{\omega}).
\end{aligned}$$

For complex α such that $\mathrm{Re}(\alpha) + \frac{m-1}{2} > 0$, we thus get in the limit $\lim\limits_{r\to 1}$, hereby making use of (4):

$$\frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}}\frac{(\alpha-k)\{(\alpha-k-1)-\underline{\omega}\epsilon(\alpha+m+k)\}P_k(\underline{\omega})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}.$$

Completely analogously, we find for $< E_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) >$:

$$\begin{aligned}
&< E_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) > \\
&\quad = -c(\alpha, m)(1+\alpha-k)\frac{\pi^{(1/2)}\Gamma(\alpha+m+k)\Gamma((m-1)/2)}{2^k\Gamma(\alpha+m)\Gamma(k+(m/2))} \\
&\qquad \times r^k\epsilon F\left(\frac{k-\alpha}{2}, \frac{1+k-\alpha}{2}; k+\frac{m}{2}; r^2\right) P_k(\underline{\omega}) + c(\alpha, m)
\end{aligned}$$

$$\times \frac{\pi^{(1/2)}\Gamma(\alpha+m+k)\Gamma((m-1)/2)}{2^{k+1}\Gamma(\alpha+m)\Gamma(1+k+(m/2))}$$
$$\times r^k \underline{\omega}; F\left(\frac{k-\alpha}{2}, \frac{1+k-\alpha}{2}; k+\frac{m}{2}; r^2\right) P_k(\underline{\omega}),$$

which for $\mathrm{Re}(\alpha)+\frac{m-1}{2}>0$ in the limit $\lim\limits_{r\to 1}$ reduces to

$$\frac{\Gamma((m-1)/2)}{8\pi^{((m-1)/2)}} \frac{(\alpha+m+k-1)\{(\alpha+m+k)-\underline{\omega}\epsilon(1+\alpha-k)\}Q_k(\underline{\omega})}{(\alpha+((m+1)/2))(\alpha+((m-1)/2))}.$$

Comparing the expressions for $< E_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) >$ and $< E_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) >$ in the limit $\lim_{r\to 1}$ with the expressions (5) and (6) for the boundary values of the photogenic Cauchy transform, we notice that:

$$\lim_{r\to 1} < E_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) >= \mathcal{C}_F^{-\alpha-m}[P_k] \uparrow (\underline{\omega}),$$
$$\lim_{r\to 1} < E_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) >= \mathcal{C}_F^{-\alpha-m}[Q_k] \uparrow (\underline{\omega}).$$

Putting $\beta=-\alpha-m$, we then make the following conclusion:

$$\lim_{r\to 1} < E_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) >= \mathcal{C}_F^{\beta}[P_k] \uparrow (\underline{\omega}),$$
$$\lim_{r\to 1} < E_\alpha(r\underline{\xi}, \underline{\omega}), Q_k(\underline{\xi}) >= \mathcal{C}_F^{\beta}[Q_k] \uparrow (\underline{\omega}).$$

Looking at the formulae for $\mathcal{C}_F^{\beta}[P_k] \uparrow (\underline{\omega})$ and $\lim\limits_{r\to 1} < E_\alpha(r\underline{\xi}, \underline{\omega}), P_k(\underline{\xi}) >$, it is immediately clear that these are identical except for the fact that the arguments of the photogenic kernel are switched, and that $\alpha \leftrightarrow \beta$.

This phenomenon was already encountered in previous papers (see e.g. [10]): the fundamental solution for the hyperbolic Dirac equation is both monogenic w.r.t. the Dirac operator in the variable $(T, \underline{X})$ acting from the left and the Dirac operator in the variable $(S, \underline{Y})$ acting from the right, provided that there is α-homogeneity in $(T, \underline{X})$ and at the same time β-homogeneity in $(S, \underline{Y})$.

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